1985 Calculus AB

1. Let f be the function given by $f(x) = \dfrac{2x-5}{x^2-4}$.

   (a) Find the domain of f.

   (b) Write an equation for each vertical and each horizontal asymptote for the graph of f.

   (c) Find $f'(x)$.

   (d) Write an equation for the line tangent to the graph of f at the point $(0, f(0))$.

Solution	Distribution of Points
$f(x) = \dfrac{2x-5}{x^2-4}$	
(a) Domain of f: all real numbers except $x = 2$ and $x = -2$, <u>or</u> $\{x \in R \mid x \neq 2 \text{ and } x \neq -2\}$ <u>or</u> $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ <u>or</u> $x \neq \pm 2$	(a) 1: for answer
(b) Asymptotes Vertical: $x = 2$, $x = -2$ Horizontal: $y = 0$	(b) 3: 1 for each correctly labeled asymptote
(c) $f'(x) = \dfrac{2(x^2-4) - 2x(2x-5)}{(x^2-4)^2}$ $= \dfrac{-2x^2+10x-8}{(x^2-4)^2} = \dfrac{-2(x-4)(x-1)}{(x^2-4)^2}$	(c) 2: for correct derivative
(d) Tangent line at $(0, f(0))$ $f(0) = \dfrac{5}{4}$ $f'(0) = -\dfrac{1}{2}$ $y - \dfrac{5}{4} = -\dfrac{1}{2}(x - 0)$ <u>or</u> $y = -\dfrac{1}{2}x + \dfrac{5}{4}$ <u>or</u> $2x + 4y = 5$	(d) 3: 1: for evaluation of $f(0)$; 1: for evaluation of $f'(0)$; 1: for correct linear equation using $f(0)$ and $f'(0)$

2. A particle moves along the x-axis with acceleration given by $a(t) = \cos t$ for $t \geqq 0$. At $t = 0$ the velocity $v(t)$ of the particle is 2 and the position $x(t)$ is 5.

(a) Write an expression for the velocity $v(t)$ of the particle.

(b) Write an expression for the position $x(t)$.

(c) For what values of t is the particle moving to the right? Justify your answer.

(d) Find the total distance traveled by the particle from $t = 0$ to $t = \frac{\pi}{2}$.

Solution	Distribution of Points
(a) $v(t) = \sin t + C$ $2 = \sin(0) + C$ $C = 2$ $v(t) = \sin t + 2$	(a) 2: { 1: for finding an antiderivative (even without constant) 1: for finding correct constant
(b) $x(t) = -\cos t + 2t + C$ $5 = -\cos(0) + 2(0) + C$ $C = 6$ $x(t) = -\cos t + 2t + 6$	(b) 2: { 1: for finding an antiderivative of the function found in part (a) (even without constant) 1: for finding correct constant
(c) Moves to right when $v(t) > 0$; i.e., $\sin t + 2 > 0$. This is true for all $t \geqq 0$ because $-1 \leqq \sin t \leqq 1$ for all t implies $0 < -1 + 2 \leqq \sin t + 2 \leqq 1 + 2$.	(c) 3: { 1: for indicating $v(t) > 0$ 1: for answer 1: for justification using range of sin or using appropriately labeled graph of sin Note: $\sin t > -2$ alone is not sufficient for justification.
(d) $x(0) = -\cos(0) + 2(0) + 6 = 5$ $x\left(\frac{\pi}{2}\right) = -\cos\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right) + 6 = \pi + 6$ Distance $= x\left(\frac{\pi}{2}\right) - x(0) = \pi + 1$	(d) 2: { 1: for evaluation of $x(t)$ from part (b) at pertinent points 1: for answer
or	or
$\int_0^{\frac{\pi}{2}} \lvert v(t) \rvert \, dt = \int_0^{\frac{\pi}{2}} \lvert \sin t + 2 \rvert \, dt$ $= \int_0^{\frac{\pi}{2}} (\sin t + 2) dt = -\cos t + 2t \Big\rvert_0^{\frac{\pi}{2}}$ $= \pi + 1$	2: { 1: for $v(t)$ from part (a) used in $\int_0^{\frac{\pi}{2}} \lvert v(t) \rvert \, dt$ 1: for answer

3. Let R be the region enclosed by the graphs of $y = e^{-x}$, $y = e^{x}$, and $x = \ln 4$.

(a) Find the area of R by setting up and evaluating a definite integral.

(b) Set up, but do not integrate, an integral expression in terms of a single variable for the volume generated when the region R is revolved about the x-axis.

(c) Set up, but do not integrate, an integral expression in terms of a single variable for the volume generated when the region R is revolved about the y-axis.

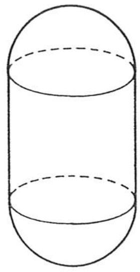
Solution	Distribution of Points
(a) $y = e^{-x}$ and $y = e^{x}$ intersect when $x = 0$. Area of $R = \int_0^{\ln 4} (e^x - e^{-x})dx$ $= (e^x + e^{-x}) \Big\vert_0^{\ln 4}$ $= \left(4 + \frac{1}{4}\right) - (1 + 1) = \frac{9}{4}$	(a) 5: 1: for finding the intersection 1: for limits of integration 1: for integrand 1: for antiderivative 1: for evaluation
(b) [Disk] $\pi \int_0^{\ln 4} (e^{2x} - e^{-2x})dx$ or [Shell] $2\pi \int_{\frac{1}{4}}^{1} y(\ln 4 + \ln y)dy +$ $2\pi \int_1^4 y(\ln 4 - \ln y)dy$	(b) 2: for correct integral expression
(c) [Disk] $\pi \int_{\frac{1}{4}}^{1} \left[(\ln 4)^2 - (\ln y)^2\right] dy +$ $\pi \int_1^4 \left[(\ln 4)^2 - (\ln y)^2\right] dy$ or $\pi \int_{\frac{1}{4}}^{4} (\ln 4)^2 dy - \pi \int_{\frac{1}{4}}^{4} (\ln y)^2 dy$ or $\frac{15}{4}\pi(\ln 4)^2 - \pi \int_{\frac{1}{4}}^{4} (\ln y)^2 dy$ or [Shell] $2\pi \int_0^{\ln 4} x[e^x - e^{-x}]dx$	(c) 2: for correct integral expression

4. Let $f(x) = 14\pi x^2$ and $g(x) = k^2 \sin\left(\frac{\pi x}{2k}\right)$ for $k > 0$.

(a) Find the average value of f on $[1, 4]$.

(b) For what value of k will the average value of g on $[0, k]$ be equal to the average value of f on $[1, 4]$?

Solution	Distribution of Points
(a) $\text{Ave}_a = \frac{1}{3}\int_1^4 14\pi x^2 dx$ $= \frac{14\pi}{3} \cdot \frac{x^3}{3}\Big\vert_1^4 = \frac{14\pi}{9}(64 - 1)$ $= 98\pi$	(a) 3: { 2: for correct integral expression 1: for antidifferentiation and evaluation }
(b) $\text{Ave}_b = \frac{1}{k}\int_0^k k^2 \sin\frac{\pi x}{2k}\, dx$ $= -\frac{2k^2}{\pi} \cos\frac{\pi x}{2k}\Big\vert_0^k$ $= \frac{-2k^2}{\pi}(0 - 1) = \frac{2k^2}{\pi}$ $\frac{2k^2}{\pi} = 98\pi$ $k^2 = 49\pi^2$ $k = 7\pi$	(b) 6: { 2: for correct integral expression 2: for antidifferentiation 1: for setting $\text{Ave}_a = \text{Ave}_b$ 1: for solving for k }

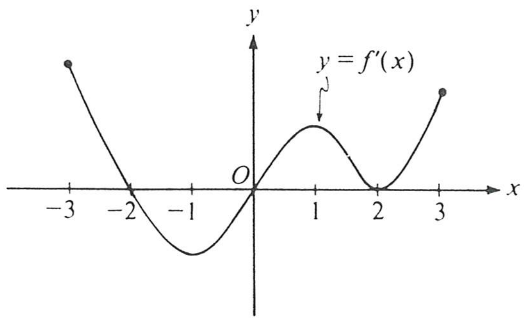


5. The balloon shown above is in the shape of a cylinder with hemispherical ends of the same radius as that of the cylinder. The balloon is being inflated at the rate of 261π cubic centimeters per minute. At the instant the radius of the cylinder is 3 centimeters, the volume of the balloon is 144π cubic centimeters and the radius of the cylinder is increasing at the rate of 2 centimeters per minute. (The volume of a cylinder with radius r and height h is $\pi r^2 h$, and the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.)

(a) At this instant, what is the height of the cylinder?

(b) At this instant, how fast is the height of the cylinder increasing?

Solution	Distribution of Points
(a) $V = \pi r^2 h + \frac{4}{3}\pi r^3$ $144\pi = \pi(3)^2 h + \frac{4}{3}\pi(3)^3$ $h = 12$	(a) 3: { 1: for correct formula 1: for substitution 1: for evaluation
(b) $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt} + 4\pi r^2 \frac{dr}{dt}$ $261\pi = \pi(3)^2 \frac{dh}{dt} + 2\pi(3)(12)(2) + 4\pi(3)^2(2)$ $\frac{dh}{dt} = 5$	(b) 6: { 4: for finding correct derivative 1: for substitution 1: for evaluation



Note: This is the graph of the derivative of f, not the graph of f.

6. The figure above shows the graph of f', the derivative of a function f. The domain of the function f is the set of all x such that $-3 \leqq x \leqq 3$.

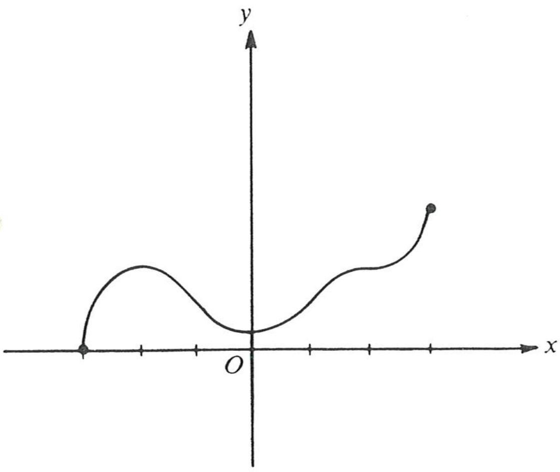
   (a) For what values of x, $-3 < x < 3$, does f have a relative maximum? A relative minimum? Justify your answer.

   (b) For what values of x is the graph of f concave up? Justify your answer.

   (c) Use the information found in parts (a) and (b) and the fact that $f(-3) = 0$ to sketch a possible graph of f on the axes provided below.

Solution	Distribution of Points
(a) f has a relative maximum at $x = -2$ Because { f' changes from positive to negative at $x = -2$ or f changes from increasing to decreasing at $x = -2$ or $f'(-2) = 0$ and $f''(-2) < 0$ } f has a relative minimum at $x = 0$ Because { f' changes from negative to positive at $x = 0$ or f changes from decreasing to increasing at $x = 0$ or $f'(0) = 0$ and $f''(0) > 0$ }	(a) 4: { 2: { 1: for indicating a relative maximum at $x = -2$ 1: for justification of correct answer } 2: { 1: for indicating a relative minimum at $x = 0$ 1: for justification of correct answer } }
(b) f is concave up on $(-1, 1)$ and $(2, 3)$ Because { f' is increasing on those intervals or $f'' > 0$ on those intervals }	(b) 2: { 1: for indicating both intervals open or closed 1: for justification of correct answer }

(c)



(c)

3: { 1: for extrema consistent with part (a)
1: for concavity consistent with part (b)
1: for $f(-3) = 0$ and a smooth curve on domain $[-3, 3]$ }